Chapter 1: Combinatorial Analysis

1.1 Introduction

A communication system is to consist of \( n \) identical antennas that are to be lined up in a linear order.

Signals are received properly as long as no two consecutive antennas are defective.

If \( m \) of the \( n \) antennas are defective, what is the probability that the resulting system will be functional?

If \( n = 4 \) and \( m = 2 \), then \( P(\text{system functional}) = \frac{3}{6} = \frac{1}{2} \).

Using counting, we can solve the problem.

If \( m = 2 \), \( n = 4 \); All possible configurations:
0110 0 means antenna is working
0101
1010
0011
1001
1100

\( P(\text{system functional}) = \frac{3}{6} = \frac{1}{2} \)

Using counting, many probability problems can be solved, just like this one.
1.1 Introduction

- The theory of counting is formally known as combinatorial analysis
- Thought/Word Problems – emphasis on logic, intuition, and creative reasoning
- Almost all problems involve abstract counting, called combinatorics
- Probability density functions, calculations need integration, summation, algebra skills.
- Often large sets of long sequences of events must be exhaustively analyzed.
- We will do most homework problems in class after you turn in.
- Quizzes based on homework problems and examples discussed in the class will be given for each week.

1.2 The Basic Principle of Counting

- Experiment 1 has $m$ possible outcomes and Experiment 2 has $n$ possible outcomes.
- If the above two experiments are to be performed, then together there are $m \times n$ possible outcomes of the two experiments
  
  $(1, 1), (1, 2), \ldots, (1, n) \\
  (2, 1), (2, 2), \ldots, (2, n) \\
  \ldots \\
  (m, 1), (m, 2), \ldots, (m, n)$

Example 2a: A small community consists of 10 women, each of whom has 3 children. If one women and one of her children are to be chosen as mother and child of the year.

How many different choices are possible? $10 \times 3 = 30$

Generalized basic principle of counting

If $r$ experiments that to be performed, and each experiment may have $n_i$ outcomes for $i = 1$ to $r$,

The total possible outcomes of these $r$ experiments is $n_1 \times n_2 \times n_3 \ldots \times n_r$

Example 2b: A college planning committee consists of 3 freshmen, 4 sophomores, 5 juniors and 2 seniors. A subcommittee of 4, consisting of 1 person from each class, is to be chosen. How many different subcommittees are possible?

$3 \times 4 \times 5 \times 2 = 120$ possible subcommittees

Example 2c:

How many different 7-place license plates are possible if the first 3 places are to be occupied by letters and the final 4 by numbers?

$26 \times 26 \times 26 \times 10 \times 10 \times 10 \times 10 = 175,760,000$

Example 2d:

How many functions defined on $n$ points are possible if each functional value is either 1 or 0?

$f(i) = 1 \text{ or } 0 \text{ for } i = 1 \text{ to } n, \implies 2^n$ possible functions.
Example 2e
How many 7-place license plates would be possible if repetition among letters or numbers were prohibited?

\[26 \times 25 \times 24 \times 10 \times 9 \times 8 \times 7 = 78,624,000\] possible license plates.

1.3 Permutations

• How many different ordered arrangements of the letters a, b, and c are possible?
  – 6 namely, abc, acb, bac, bca, cab, cba
• Each ordered arrangement is known as a permutation.
• There are 6 possible permutations of a set of 3 objects.
  \[6 = 3!\]
• If there are \(n\) objects, the total of permutations is \(n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1 = n!\)

Example 3a: How many different batting orders are possible for a baseball team consisting of 9 players?
There are 9! possible batting orders.

Example 3b
A class in probability theory consists of 6 men and 4 women. An examination is given and students are ranked according to their performance. Assume that no two students obtain the same score.

(a) How many different rankings are possible?
\[10!\]

(b) If the men are ranked among just themselves and the women themselves, how many different rankings are possible?
\[(6!) \times (4!) = 17,280.\]

Example 3c
10 books. Of these, 4 are math books, 3 are chemistry books, 2 are history books and 1 is a language book. same subject are together on the shelf. How many different arrangements are possible?

\[4! \times 3! \times 2! \times 1! = 6912.\]

Example 3d: How many different letter arrangements can be formed using the letters P E P P E R?

In general \(n\) objects of which \(n_1\) are alike, \(n_2\) are alike, \(n_r\) are alike, the total different permutations are:

\[\frac{n!}{n_1! n_2! \cdots n_r!}; \text{where}, n = n_1 + n_2 + \cdots + n_r\]
Example 3e:
A chess tournament has 10 competitors of which 4 are Russian, 3 are from the United States, 2 from Great Britain, and 1 from Brazil. If the tournament result lists just the nationalities of the players in the order in which they place, how many outcomes are possible?

$$\frac{10!}{4!3!2!1!} = 12,600$$

Example 3f
How many different signals, each consisting of 9 flags hung in a line, can be made from a set of 4 white flags, 3 red flags, and 2 blue flags if all flags of the same color are identical?

There are $9! / (4!3!2!) = 1260$.

1.4 Combinations

- Determine the number of different groups of $r$ objects that could be selected from $n$ objects. (the order of selection is irrelevant, such as abc, acb counts as one group.)

$$C(n, r) = \frac{n!}{(n-r)! r!} = \binom{n}{r}$$

Example 4a
n = 20, r = 3, such as select 3 person to serve in a committee from 20 people. The number of possible committees $C(20, 3) = 20 \times 19 \times 18 / (3 \times 2 \times 1) = 1140$

Example 4b: For a group of 5 women and 7 men, how many committees of 2 women and 3 men can be formed?

Solution: $C(5, 2) \times C(7, 3) = 350$.

What if 2 of the men are feuding and refuse to serve on the same committee together?

Solution: calculating the two feuding men in the same committees then subtract from the total available men groups.

$C(2, 2) \times C(5, 1) = 5$ out of $C(7, 3) = 35$. 35 - 5 = 30 for men
$C(5, 2) = 10$ for women, total 30 x 10 = 300 committees.

Example 4c
Consider a set of $n$ antennas of which $m$ are defective and $n-m$ are functional and assume that all the defectives and all of the functionals are considered indistinguishable. How many linear orderings are there in which no two defectives are consecutive?

$n-m$ functional antennas. There are $n-m+1$ possible positions to put the defective antennas such that at least one functional antenna between any two defective ones.

$\ ^1 \ ^1 \ ^1 \ ^1 \ ^1 \ \ldots \ldots \ , \ 1; \ 1 = functional; \ ^1 = place \ for \ at \ most \ one \ defective$

There are $C(n-m+1, m)$ possible orderings

if $n= 4, m= 2, then C(3, 2) = 3$
The Binomial Theorem

\[ \binom{n}{r} \] are often referred to as binomial coefficients. This is because of their prominence in the binomial theorem.

\[
(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{This is binomial theorem}
\]

\[
\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r} \quad \text{A useful combinatorial identity}
\]

### 1.5 Multinomial Coefficient

- A set of \( n \) distinct items is to be divided into \( r \) distinct groups of respective sizes \( n_1, n_2, n_3, \ldots, n_r \), where \( n = n_1 + n_2 + n_3 + \ldots + n_r \); the number of possible divisions is \( \frac{n!}{n_1!n_2!\cdots n_r!} \).

- A police station of 10 officers, 5 patrolling the street, 2 working in the station, 3 on reserve, How many different divisions of the 10 officers into 3 groups are possible? \( \frac{10!}{5!2!3!} = 2520 \)

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ECE 3610

Name: ___________________(print)

Quiz: 2006.01.19

How many different 7-place license plates are possible if the first three places are for numbers and the other 4 for letters?